UNIONS OF INCREASING AND INTERSECTIONS OF DECREASING SEQUENCES OF CONVEX SETS

BY

VICTOR KLEE

ABSTRACT

We prove that a convex set C is a polytope if and only if C is not the union of any strictly increasing sequence of convex sets. In addition, we attempt (with partial success) to characterize, in intrinsic geometric terms, those convex subsets C of a convex set X such that C is not the intersection of any strictly decreasing sequence of convex subsets of X.

Introduction

For a convex set X in a real vector space, let $\boldsymbol{C}(X)$ denote the class of all convex subsets of X, $\boldsymbol{U}(X)$ the class of all unions of strictly increasing sequences in $\boldsymbol{C}(X)$, and $\boldsymbol{I}(X)$ the class of all intersections of strictly decreasing sequences in $\boldsymbol{C}(X)$. Let $\boldsymbol{U}'(X) = \boldsymbol{C}(X) \sim \boldsymbol{U}(X)$ and $\boldsymbol{I}'(X) = \boldsymbol{C}(X) \sim \boldsymbol{I}(X)$. Though not very natural from a purely geometric viewpoint, the consideration of these subclasses of $\boldsymbol{C}(X)$ does arise naturally in studying the complete lattice formed by $\boldsymbol{C}(X)$ with respect to set-theoretic inclusion. The purpose of the present note is to describe the subclasses $\boldsymbol{U}(X)$ and $\boldsymbol{I}(X)$ (or, equivalently, $\boldsymbol{U}'(X)$ and $\boldsymbol{I}'(X)$) in geometric terms, especially when X is a finite-dimensional flat. The main results are stated below.

THEOREM 1. *A convex set is a polytope if and only if it is not the union of any strictly increasing sequence of convex sets.*

THEOREM 2. *If X is a finite-dimensional convex set and $C \in \boldsymbol{I}'(X)$, then the closure of C is the intersection of a polyhedron with the closure of X. If X is a quasipolyhedron then so is C.*

Polytopes, polyhedra, and quasipolyhedra are defined in the next section.

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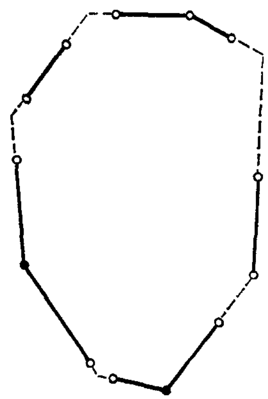
THEOREM 3. *If X is a finite-dimensional flat and $C \in \boldsymbol{C}(X)$, then $C \in \boldsymbol{I}'(X)$ if and only if every class in $\boldsymbol{C}(X)$ whose intersection is C admits a finite subclass with the same intersection.*

THEOREM 4. *A subset of the plane R^2 belongs to $\boldsymbol{I}'(R^2)$ if and only if it is of the form $Q \cup S \cup V$, where Q is an open convex n-gon (not necessarily bounded), S is the union of n open segments or rays properly contained in the respective open edges of Q, and each point of V is a vertex of Q which is an endpoint of two of the segments or rays forming S.*

The figure below depicts a typical member of $\boldsymbol{I}'(R^2)$.



A constructive characterization of $\boldsymbol{I}'(R^3)$ is also obtained. It is stated as Theorem 5 and appears at the end of the paper.

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Definitions

As the terms are used here, a *polyhedron* is the intersection of a finite number of closed halfspaces in a finite-dimensional real vector space, while a *polytope* is the convex hull of a finite set of points. It is known that the polytopes are precisely the bounded polyhedra. A *face* of a convex set C is a convex subset F of C such that F contains the closed segment $[x, y]$ whenever x and y are points of C for which F intersects the open segment $]x, y[$. The $(d-1)$-dimensional faces of a d-dimensional polyhedron are called its *facets*. It is known that the polyhedra are precisely the finite-dimensional closed convex sets which have only finitely many faces, and that each face of a polyhedron is a polyhedron. For the elementary aspects of facial structure and polyhedra which are used here without explicit reference, see the introductory portions of [1] and [4].

A convex set is here called a *quasipolyhedron* provided that the closure of each of its faces is a polyhedron. Note that this applies to the set itself, as it is a face of itself. It is easily seen that a quasipolyhedron has only finitely many faces, each of which is a quasipolyhedron (Proposition 3).

When X is a convex subset of a finite-dimensional real vector space, the *relative interior* of X, denoted by rint X, is the interior of X relative to the smallest flat containing X. The set $\operatorname{rint} X$ is nonempty whenever X is nonempty, convex and finite-dimensional. The *relative boundary* of X, denoted by rbd X, is the set $(\operatorname{cl} X) \sim (\operatorname{rint} X)$. The relative interior of a polyhedron P is called an *open polyhedron*, and the relative interiors of P's faces are the *open faces* of $\operatorname{rint} P$.

A *semispace* in a flat A is a set which, for some point p of A, is a maximal convex subset of $A \sim \{p\}$. The basic references on semispaces are [2] and [3].

Proof of Theorem 1

Let C be a convex subset of a real vector space. If C is a polytope then C is the convex hull $\operatorname{con} Y$ of a finite set Y. For any increasing sequence $C_1 \subset C_2 \subset \cdots$ of convex sets whose union is C, there exists m such that $C_m \supset Y$. But then $C_n = C$ for all $n \geqq m$, and the sequence is not strictly increasing. That settles the "only if" part.

For the "if" part, let us assume that C is not the union of any strictly increasing sequence of convex sets. It is easily seen that C is finite-dimensional and bounded, whence $\operatorname{cl} C$ is compact. For each extreme point p of $\operatorname{cl} C$ there is (by a well-known result) a sequence $Q_1, Q_2, \cdots$ of closed halfspaces such that $(Q_n \cap \operatorname{cl} C)_{n=1}^{\infty}$ is a strictly decreasing sequence whose intersection is $\{p\}$. But then $p \in C$, for otherwise $(C \sim Q_n)_{n=1}^{\infty}$ is a strictly increasing sequence of convex sets whose union is C. With $\operatorname{ext} C \subset C$, we have

$$C \subset \operatorname{cl} C = \operatorname{con} \operatorname{ext} \operatorname{cl} C \subset \operatorname{con} \operatorname{ext} C \subset C,$$

where the equality follows from another well-known result. But then $C = \operatorname{con} \operatorname{ext} C$, and to complete the proof it suffices to observe that $\operatorname{ext} C$ is finite. Indeed, if $p_1, p_2, \cdots$ are distinct extreme points of C and if $C_n = C \sim \{p_n, p_{n+1}, \cdots\}$, then $(C_n)_{n=1}^{\infty}$ is a strictly increasing sequence of convex sets whose union is C.

COROLLARY *For any convex set X the class $U(X) \cup I(X)$ is equal to*

$$\begin{cases} C(X) \text{ when } X \text{ is not a polytope} \\ C(X) \sim \{X\} \text{ when } X \text{ is a polytope of dimension } \geqq 1 \\ \emptyset \text{ when } X \text{ is empty or consists of a single point.} \end{cases}$$

PROOF. Use Theorem 1 in conjunction with the fact that if $\dim X \geqq 1$ then $P \in I(X)$ for every polytope P contained in and not equal to X.

Proof of Theorem 2

With $K = X$ in condition (a_1), the following result settles the first part of Theorem 2. With $K = X$ in condition (a_2), it is used in proving the second part of Theorem 2.

PROPOSITION 1. *Suppose that C and K are finite-dimensional convex sets, that*

(a_1) $C \subset K$

or

(a_2) *K is a maximal convex subset of the relative boundary of a face F of C, and that*

(b) *C is not the intersection of any strictly decreasing sequence of convex subsets of $C \cup \operatorname{rint} K$.*

Then

(c) *the set $\operatorname{cl}(C \cap K)$ is the intersection of $\operatorname{cl} K$ with a polyhedron.*

Further,

(d) $\dim(C \cap K) = \dim K$

if $\dim K \geqq 1$, (b) *holds, and either* (a_1) *holds or* (a_2) *holds with $F = C$.*

PROOF. We first show that (d) holds under the stated hypotheses. It is easily verified that these hypotheses imply the existence of a point $c \in C \cap K$. Suppose that (d) fails, whence the set $C \cap K$ lies in a flat H which does not contain $\operatorname{rint} K$. Choose $k \in (\operatorname{rint} K) \sim H$ and for each n let

$$C_n = \operatorname{con}\left(C \cup \left\{ \frac{n-1}{n} c + \frac{1}{n} k \right\} \right).$$

It is easily verified that

$$C_n = C \cup \operatorname{con}\left((C \cap K) \cup \left\{ \frac{n-1}{n} c + \frac{1}{n} k \right\}\right) \subset C \cup \operatorname{rint} K$$

and that $(C_n)_{n=1}^{\infty}$ is a strictly decreasing sequence of convex sets whose intersection is C. The contradiction yields the desired conclusion.

In establishing (c), we may assume that $\dim(C \cap K) \geqq 2$. Let $\mathbf{M}$ denote the class of all maximal convex subsets of $\operatorname{rbd}(C \cap K)$, and note that $(\operatorname{rint} M_1) \cap (\operatorname{rint} M_2) = \emptyset$ for any two distinct members M_1 and M_2 of $\mathbf{M}$. Let $\mathbf{M}'$ denote the class of all $M \in \mathbf{M}$ such that M intersects rint K, and note that $\operatorname{rint} K \supset \operatorname{rint} M$ for all $M \in \mathbf{M}'$. Let $\mathbf{M}''$ denote the class of all $M \in \mathbf{M}'$ for which $C \not\supset \operatorname{rint} M$. We claim that the class $\mathbf{M}''$ is finite. Indeed, if $M_1, M_2, \cdots$ is a sequence of distinct members of $\mathbf{M}''$ and if

$$C_n = C \cup \Big(\bigcup_{i=n}^{\infty} \operatorname{rint} M_i\Big),$$

then $(C_n)_1^{\infty}$ is a strictly decreasing sequence of subsets of $C \cup \operatorname{rint} K$ whose intersection is C. Further, each set C_n is convex, and the resulting contradiction implies that $\mathbf{M}''$ is finite.

Since each summand of C_n is convex, it suffices in proving the convexity of C_n to show that if $x \in \operatorname{rint} M_i$ with $i \geqq n$, then $]x, y[\, \subset C_n$ for all $y \in C_n$. When $y \in \operatorname{cl} M_i$ we have $]x, y[\, \subset \operatorname{rint} M_i \subset C_n$. Suppose, on the other hand, that $y \in C_n \sim \operatorname{cl} M_i$. If (a_1) holds then $]x, y[\, \subset \operatorname{rint} C \subset C_n$. If (a_2) holds and $y \notin K$ then

$$]x, y] \subset (\operatorname{rint} C) \cup (\operatorname{rint} F) \subset C_n.$$

In the remaining case, (a_2) holds and $y \in (C \cap K) \sim \operatorname{cl} M_i$, whence

$$]x, y[\, \subset \operatorname{rint}\,(C \cap K) \subset C_n.$$

Having shown that $\mathbf{M}''$ is finite, we define

$$B = (\operatorname{rbd}(C \cap K)) \cap (\operatorname{rint} K)$$

and will proceed to show that $B \subset \cup\, \mathbf{M}''$. Supposing the contrary, let

$$U = B \sim \cup\, \mathbf{M}'' \neq \emptyset.$$

Note that U is open relative to B and that every member of $\mathbf{M}$ intersecting U has its entire relative interior contained in C. Choose a member M_0 of $\mathbf{M}$ which intersects U, and choose a point $u \in \operatorname{rint} M_0$. Choose $c \in \operatorname{rint}(C \cap K)$, and for each n let

$$k_n = -\frac{1}{n}c + \frac{n+1}{n}u \quad \text{and} \quad C_n = \operatorname{con}(C \cup \{k_n\}),$$

so that $(C_n)_{n=1}^{\infty}$ is a strictly decreasing sequence of convex sets. As $u \in \operatorname{rint} K$, it is true for all sufficiently large n that $k_n \in \operatorname{rint} K$ and thus $C_n \subset C \cup \operatorname{rint} K$. By hypothesis, then there exists a point

$$q \in (\bigcap_{n=1}^{\infty} C_n) \sim C.$$

Plainly $q \in \operatorname{rint} K$. For each n there exist $c_n \in C$ and $\lambda_n \in [0,1]$ such that

$$q = (1 - \lambda_n)c_n + \lambda_n k_n ,$$

whence the entire segment $[q, q_n]$ misses C and it follows that

$$[q, u] \subset \operatorname{cl}(C \cap K) \sim \operatorname{rint}(C \cap K) .$$

Let aff$K$ denote the smallest flat containing K. Then there is a flat H of deficiency 1 in aff K which supports the convex set $C \cap K$ at the point $(q/2) + (u/2)$. The intersection $C \cap H$ is a convex subset of $\operatorname{rbd}(C \cap K)$ which contains $[q, u]$, and with $u \in \operatorname{rint} M_0$ it follows that $C \cap H \supset \{q\} \cup M_0$. That contradicts the maximality of M_0 and shows that $B \subset \cup M''$.

Knowing, now, that M'' is finite and $B \subset \cup M''$, we proceed to choose, for each $M \in M''$, a closed halfspace Q_M in aff$K$ such that Q_M contains $C \cap K$ and the bounding hyperplane of Q_M contains M. Then

$$\operatorname{cl}(C \cap K) = (\operatorname{cl} K) \cap (\textstyle\bigcap_{M \in M''} Q_M).$$

To establish this, note that if $c_0 \in \operatorname{rint}(C \cap K)$, $k \in (\operatorname{rint} K) \sim \operatorname{rint}(C \cap K)$, p is the point at which the boundary of C is intersected by the segment $]c_0, k[$, and M is a member of M'' which includes p, then $k \notin Q_M$. That completes the proof of Proposition 1.

A slight modification of the above argument can be used to prove the following:

PROPOSITION 2. *Suppose that K is a convex subset of a topological linear space E, and that $C \in I'(K)$ with* $\operatorname{int} C \neq \emptyset$. *Suppose that every maximal convex subset of* $\operatorname{cl} C$ *which intersects* $\operatorname{int} K$ *is either finite-dimensional or has nonempty interior relative to a flat of finite deficiency in E. Then* $\operatorname{cl} C$ *is the intersection of* $\operatorname{cl} K$ *with a finite number of closed halfspaces.*

To complete the proof of Theorem 2 we must show that if X is a quasipolyhedron and $C \in I'(X)$, then C is a quasipolyhedron. Suppose that X is d-dimensional, and for $0 \leqq k \leqq d$ let A_k denote the assertion that each k-dimensional face of C has polyhedral closure. Then A_0 and A_1 are trivially correct, while A_d follows from Proposition 1, using (a_1) with $K = X$. Now consider a k with $1 < k < d$, supposing that A_l is known whenever $k < l \leqq d$, and consider an arbitrary k-dimensional face G of C. Let F denote the smallest face of C that properly contains G, whence $\mathrm{cl}F$ is a polyhedron by the inductive hypothesis. Let K denote the facet of $\mathrm{cl}F$ which contains G. It then follows from Proposition 1, using (a_2), that the set $\mathrm{cl}(C \cap K)$ is the intersection of $\mathrm{cl}K$ whith a polyhedron. As $\mathrm{cl}(C \cap K) = \mathrm{cl}G$, while K is itself a polyhedron, we conclude that $\mathrm{cl}G$ is a polyhedron and that A_k holds.

The proof of Theorem 2 is now complete.

Proof of Theorem 3

Only the last conclusion of Proposition 3 will be used in proving Theorem 3.

PROPOSITION 3. *If X is a quasipolyhedron in a finite-dimensional flat A, then X has only finitely many faces and each face of X is a quasipolyhedron. Further, X is the intersection of a countable number of semispaces in A.*

PROOF. For the first assertion, let us suppose that X has infinitely many faces and let G be of minimum dimension among those faces of X which contain infinitely many faces of X. The set $\mathrm{cl}\,G$ is a polyhedron and, with the exception of G itself, each face of X that is contained in G lies in some facet of $\mathrm{cl}\,G$. For each facet F of $\mathrm{cl}\,G$, the intersection $F \cap G$ is a face of X which, by the minimality of G, contains only finitely many faces of X. As $\mathrm{cl}\,G$ has only finitely many facets, it follows that G contains only finitely many faces of X. That is a contradiction showing that the number of faces of X is finite. Since any face of a face of X is itself a face of X, it follows immediately from the definition of quasipolyhedron that any face of a quasipolyhedron is itself a quasipolyhedron.

The second part of Proposition 3 is obvious when the dimension of the flat is 1. Suppose that it is known in the $(d-1)$-dimensional case, and consider a quasipolyhedron X in a d-dimensional flat A. Let J denote the smallest flat containing X. If $J \neq A$, the inductive hypothesis implies that X is the intersection of a countable number of semispaces in J. Further, each such semispace is the intersection with J of a semispace in A, and J itself is the intersection of a countable number of semispaces in A. It follows, in this case, that X is the intersection of a countable number of semispaces in A. In the remaining case, $J = A$ and we denote by $F_1, \cdots, F_n$ the facets of the polyhedron $\mathrm{cl}X$. For each i, let H_i and Q_i denote respectively the hyperplane containing F_i and the open halfspace which is bounded by H_i and contains the interior of X. For each i, the set $X \cap H_i$ is a quasipolyhedron and hence by the inductive hypothesis is the intersection of a countable sequence S_1^i, $S_2^i, \cdots$, of semispaces in H_i. With $T_j^i = S_j^i \cup Q_i$, the T_j^i's ($i = 1, \cdots, n$, $j = 1, 2, \cdots$) form a countable class of semispaces whose intersection is X. That completes the proof of Proposition 3.

The "if" part of Theorem 3 is obvious. For the "only if" part, we want to show that if $C \in \boldsymbol{I}'(E)$ and if $\boldsymbol{K}$ is a class in $\boldsymbol{C}(E)$ whose intersection is C, then $\boldsymbol{K}$ admits a finite subclass with intersection C. Note that C is a quasipolyhedron by Theorem 2, and hence by Proposition 3 is the intersection of a countable number of semispaces. It follows from Theorem 3.2 of [3] that C is the intersection of a countable subclass $\boldsymbol{K}'$ of $\boldsymbol{K}$, and then from the membership of C in $\boldsymbol{I}'(E)$ that $\boldsymbol{K}'$ admits a finite subclass whose intersection is C.

Proof of Theorem 4 and statement of Theorem 5

If E is a d-dimensional real vector space, with $d \geqq 2$, and $C \in \boldsymbol{I}'(E)$, then:

(i) C is a d-dimensional quasipolyhedron;

(ii) for each facet F of the polyhedron $\mathrm{cl}\, C$, the intersection $C \cap F$ is a $(d-1)$-dimensional quasipolyhedron;

(iii) C does not contain the entire relative interior of any facet of $\mathrm{cl}\ C$.

Assertions (i) and (ii) follow from Proposition 1, while (iii) is an easily verified direct consequence of the fact that C is not the intersection of any strictly decreasing sequence of convex subsets of E.

When $d = 2$ it follows from the preceding paragraph that each member C of $\boldsymbol{I}'(R^2)$ is of the form $Q \cup S \cup V$, where Q is an open convex n-gon, S is the union of n open segments or rays properly contained in the respective open edges of Q, and each point of V is an endpoint of a segment or ray forming S. In fact, either C's convexity or C's membership in $\boldsymbol{I}'(R^2)$ is violated unless V is precisely as described in Theorem 3. Finally, it is a routine matter to verify that the sets described in Theorem 3 are all members of $\boldsymbol{I}'(R^2)$. (The n-gons are, of course, not required to be bounded. In fact, the term *convex n-gon* in Theorem 2 means a 2-dimensional polyhedron having n facets. That is a plane when $n = 0$, a halfplane when $n = 1$, and a plane angle or the strip between two parallel lines when $n = 2$.)

A similar (but considerably more detailed) argument yields the following result.

THEOREM 5. *A subset C of R^3 belongs to $\boldsymbol{I}'(R^3)$ if and only if C can be expressed in the form,*

$$C = P \cup Q \cup S \cup V \cup T \cup W,$$

where the summands are obtained in the following way:

P is an open 3-dimensional polyhedron with m open facets $F_1, \cdots, F_m$;

Q is the union of m open convex polygons $Q_1, \cdots, Q_m$, where Q_i is properly contained in F_i;

let $J_1^i, \cdots, J_{n(i)}^i$ denote the open edges of Q_i which lie in F_i and let $n = \sum_{i=1}^m n(i)$; let $K_1, \cdots, K_r$ denote the 1-dimensional sets obtained by intersecting an edge of P with the closures of Q_i and Q_i^1, where F_i and F_i^1 are the facets of P incident to the edge in question; let K_0 denote the union of all onepointed sets obtained in the same way;

S is the union of n open segments or rays $S_1^1, \cdots, S_{n(m)}^m$, where S_h^i is properly contained in J_h^i;

each point V is a vertex of some Q_i, belongs to the corresponding F_i, and is an endpoint of two of the open segments or rays S_h^i;

T is the union of r open segments or rays $T_1, \cdots, T_r$, where T_l is contained in (and may be equal to) $\mathrm{rint}\, K_l$; *let Y denote the set af all points y such that, for some l, y is an endpoint of both T_l and K_l;*

each point of W is a vertex of P or belongs to $K_0 \cup Y$, the points of W being chosen so as not to disturb the convexity of C.

A more detailed description of W could be provided, but it would render the statement of Theorem 5 even more complicated.

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University of Washington
and
University of Colorado